

## HYPERSTABILITY OF A GENERAL QUINTIC FUNCTIONAL EQUATION AND A GENERAL SEPTIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we present some hyperstability results for a general quintic functional equation and a general septic functional equation.

### 1. Introduction

Throughout this paper, let  $V$ ,  $X$  and  $Y$  be a real vector space, a real normed space, and a real Banach space, respectively. For a mapping  $f$  from  $V$  to  $Y$ , we consider the functional equation

$$(1.1) \quad \sum_{i=0}^n {}_n C_i (-1)^{n-i} f(x + iy) = 0,$$

where  $n \in \mathbb{N}$ . Observe that a solution mapping  $f : V \rightarrow Y$  of (1.1) is a generalized polynomial mapping of degree at most  $n - 1$  in the sense of J. Baker in [1]. So, for  $n = 2, 3, 4, 5, 6, 7$ , and  $8$ , we call the functional equation (1.1) a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic functional equation, respectively.

Recall the stability problem of a functional equation was started by S. M. Ulam[14] and D. H. Hyers[4], and many mathematicians have generalized by showing the stability results of various kind of functional equations (see [1, 2, 5, 6, 11, 13], etc.). Prior to this paper, for the functional equation (1.1), with  $n = 3, 4, 5, 6$ , and  $7$ , stability problems were studied by Y.-H. Lee[7, 8, 9, 10], by following;

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**THEOREM 1.1.** [10] *Let  $p \neq 1, 2, 3, 4, 5$  be a fixed nonnegative real number and  $\theta > 0$  be a real number. Suppose that  $f : X \rightarrow Y$  is a mapping such that*

$$(1.2) \quad \left\| \sum_{i=0}^6 {}_6C_i(-1)^{6-i} f(x + iy) \right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ . Then there exists a solution mapping  $F$  of the functional equation

$$\sum_{i=0}^6 {}_6C_i(-1)^{6-i} F(x + iy) = 0$$

such that  $\|f(x) - f(0) - F(x)\|$  is less than or equal to

$$\left\{ \begin{array}{ll} \left( \frac{K}{45 \cdot 2^{2p}(2^p-2)} + \frac{(128+44 \cdot 2^p)K}{45(2^p-32)(2^p-8)2^{2p}} + \frac{8}{2^p(2^p-16)(2^p-4)} \right) \theta \|x\|^p & \text{if } 5 < p, \\ \left( \frac{(2 \cdot 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{8}{2^p(2^p-16)(2^p-4)} \right) \theta \|x\|^p & \text{if } 4 < p < 5, \\ \left( \frac{(2 \cdot 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{8}{(2^p-4)(16-2^p)} \right) \theta \|x\|^p & \text{if } 3 < p < 4, \\ \left( \frac{K}{90 \cdot 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{8}{(2^p-4)(16-2^p)} \right) \theta \|x\|^p & \text{if } 2 < p < 3, \\ \left( \frac{K}{90 \cdot 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{8}{(16-2^p)(4-2^p)} \right) \theta \|x\|^p & \text{if } 1 < p < 2, \\ \left( \frac{(22+2^p)K}{720(8-2^p)(2-2^p)} + \frac{K}{720(32-2^p)} + \frac{8}{(16-2^p)(4-2^p)} \right) \theta \|x\|^p & \text{if } 0 \leq p < 1 \end{array} \right.$$

for all  $x \in X \setminus \{0\}$  and  $F(0) = 0$ , where  $K = 182 + 38 \cdot 2^p + 6 \cdot 3^p$ .

In this article, we present some hyperstability results for the general quintic functional equation

$$(1.3) \quad Cf(x, y) := \sum_{i=0}^6 {}_6C_i(-1)^{6-i} f(x + iy) = 0$$

and the general septic functional equation

$$(1.4) \quad Df(x, y) := \sum_{i=0}^8 {}_8C_i(-1)^{8-i} f(x + iy) = 0,$$

which are special parts of the study of stability problems of functional equations as proving a solution of a functional equation differing slightly from the given equation is exactly the solution. Precisely, let  $p < 0$  and

$\theta > 0$  be real numbers, and let the mapping  $f : X \rightarrow Y$  satisfy the inequality

$$\|Cf(x, y)\|, \|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X \setminus \{0\}$ , then we will show that  $f$  must be a solution of

$$Cf(x, y) = 0, \quad Df(x, y) = 0$$

for all  $x, y \in X$ , respectively. A solution mapping of the general quintic functional equation is called a general quintic mapping and the solution mapping. On the other hand, we call a solution of the general septic functional equation a general septic mapping.

Moreover, the hyperstability problems of the other type functional equations were studied by E. Gselmann[3], and A. Najati et al.[12].

## 2. Hyperstability of a general quintic functional equation

Throughout this paper, for  $f : V \rightarrow Y$ , we use the following abbreviations,

$$\tilde{f}(x) := f(x) - f(0), \quad f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}$$

for all  $x \in V$ .

LEMMA 2.1. For a given mapping  $f : V \rightarrow Y$ , let  $f_1, f_2, f_3, f_4, f_5 : V \rightarrow Y$  be the mappings defined by following;

$$\begin{aligned} f_1(x) &:= \frac{1}{L} \begin{vmatrix} f_o(x) & 1 & 1 \\ f_o(2x) & 8 & 32 \\ f_o(4x) & 8^2 & 32^2 \end{vmatrix} = \frac{4(256f_o(x) - 40f_o(2x) + f_o(4x))}{720}, \\ f_2(x) &:= \frac{1}{L'} \begin{vmatrix} f_e(x) & 1 \\ f_e(2x) & 16 \end{vmatrix} = \frac{16f_e(x) - f_e(2x)}{12}, \\ f_3(x) &:= \frac{1}{L} \begin{vmatrix} 1 & f_o(x) & 1 \\ 2 & f_o(2x) & 32 \\ 2^2 & f_o(4x) & 32^2 \end{vmatrix} = -\frac{5(64f_o(x) - 34f_o(2x) + f_o(4x))}{720}, \\ f_4(x) &:= \frac{1}{L'} \begin{vmatrix} 1 & f_e(x) \\ 4 & f_e(2x) \end{vmatrix} = -\frac{4f_e(x) - f_e(2x)}{12}, \\ f_5(x) &:= \frac{1}{L} \begin{vmatrix} 1 & 1 & f_o(x) \\ 2 & 8 & f_o(2x) \\ 2^2 & 8^2 & f_o(4x) \end{vmatrix} = \frac{16f_o(x) - 10f_o(2x) + f_o(4x)}{720} \end{aligned}$$

for all  $x \in V$ , where  $L := \begin{vmatrix} 1 & 1 & 1 \\ 2 & 8 & 32 \\ 2^2 & 8^2 & 32^2 \end{vmatrix}$  and  $L' := \begin{vmatrix} 1 & 1 \\ 4 & 16 \end{vmatrix}$ . Then we can show that  $f_o(x) = f_1(x) + f_3(x) + f_5(x)$  and  $f_e(x) = f_2(x) + f_4(x)$ , and so

$$(2.1) \quad f(x) = f_o(x) + f_e(x) = \sum_{i=1}^5 f_i(x)$$

for all  $x \in V$ .

**Proof.** We note that  $L, L' \neq 0$ . By the uniqueness of solution (stated in Cramer's rule), the family  $\{f_1(x), f_3(x), f_5(x)\}$  is the only solution to the system of nonhomogeneous linear equations

$$(2.2) \quad \begin{cases} f_1(x) + f_3(x) + f_5(x) = f_o(x), \\ 2f_1(x) + 8f_3(x) + 32f_5(x) = f_o(2x), \\ 2^2f_1(x) + 8^2f_3(x) + 32^2f_5(x) = f_o(4x) \end{cases}$$

for all  $x \in V$ . It follows from the first equation of (2.2) that  $f_o(x) = f_1(x) + f_3(x) + f_5(x)$  for all  $x \in V$ . Similarly, we can show that  $f_e(x) = f_2(x) + f_4(x)$  for all  $x \in V$ . □

Moreover, in this section, we use the following definitions.

**DEFINITION 2.2.** For a given mapping  $f : V \rightarrow Y$ , we define the mappings  $Cf : V^2 \rightarrow Y$  and  $\Gamma f, \Delta f : V \rightarrow Y$  as following;

$$Cf(x, y) := \sum_{i=0}^6 {}_6C_i (-1)^{6-i} f(x + iy),$$

$$\Gamma f(x) := Cf_o(-4x, 2x) + 6Cf_o(6x, -x) + 36Cf_o(-x, x) + 70Cf_o(-2x, x),$$

$$\Delta f(x) := Cf_e(-2x, x) + 3Cf_e(-3x, x)$$

for all  $x, y \in V$ .

By laborious computation, we can get the equalities in the following lemma.

**LEMMA 2.3.** Let  $f : V \rightarrow Y$  be an arbitrarily given mapping. Then the equalities

$$(2.3) \quad \Gamma \tilde{f}(x) = \Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x),$$

$$(2.4) \quad \Delta \tilde{f}(x) = \tilde{f}_e(4x) - 20\tilde{f}_e(2x) + 64\tilde{f}_e(x)$$

hold for all  $x \in V$ .

Now we will prove the hyperstability of the quintic functional equation  $Cf(x, y) = 0$  by following;

**THEOREM 2.4.** *Let  $p < 0$  and  $\theta > 0$  be real numbers. Suppose that  $f : X \rightarrow Y$  is a mapping such that*

$$(2.5) \quad \|Cf(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X \setminus \{0\}$ . Then  $Cf(x, y) = 0$  for all  $x, y \in X$ , and so  $f$  is a generalized polynomial mapping of degree at most 5.

**Proof.** Since  $\tilde{f}(0) = 0$  and  $C\tilde{f}(x, y) = Cf(x, y)$  for all  $x, y \in X$ , by (2.5) we get

$$(2.6) \quad \|C\tilde{f}(x, y)\|, \|C\tilde{f}_o(x, y)\|, \|C\tilde{f}_e(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X \setminus \{0\}$ . Together with the definitions of  $\Gamma\tilde{f}$  and  $\Delta\tilde{f}$  in Definition 2.1, it follows that

$$(2.7) \quad \|\Gamma\tilde{f}(x)\| \leq 720K'\theta\|x\|^p,$$

$$(2.8) \quad \|\Delta\tilde{f}(x)\| \leq 12K\theta\|x\|^p$$

for all  $x \in X \setminus \{0\}$ , where

$$K' := \frac{6 \cdot 6^p + 4^p + 71 \cdot 2^p + 148}{720} \quad \text{and} \quad K := \frac{3 \cdot 3^p + 2^p + 4}{12}.$$

Now, using the definitions of  $f_1, f_2, \dots$ , and  $f_5$  in Lemma 2.1, together with the properties (2.3), (2.4), (2.7) and (2.8), we can show the following inequalities

$$\begin{aligned} \left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right\| &= \left\| -\frac{4\Gamma\tilde{f}(2^i x)}{720 \cdot 2^{i+1}} \right\| \leq \frac{4K'\theta\|2^i x\|^p}{2^{i+1}}, \\ \left\| \frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1} x)}{4^{i+1}} \right\| &= \left\| \frac{\Delta\tilde{f}(2^i x)}{12 \cdot 4^{i+1}} \right\| \leq \frac{K\theta\|2^i x\|^p}{4^{i+1}}, \\ \left\| \frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1} x)}{8^{i+1}} \right\| &= \left\| \frac{5\Gamma\tilde{f}(2^i x)}{720 \cdot 8^{i+1}} \right\| \leq \frac{5K'\theta\|2^i x\|^p}{8^{i+1}}, \\ \left\| \frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1} x)}{16^{i+1}} \right\| &= \left\| -\frac{\Delta\tilde{f}(2^i x)}{12 \cdot 16^{i+1}} \right\| \leq \frac{K\theta\|2^i x\|^p}{16^{i+1}}, \\ \left\| \frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1} x)}{32^{i+1}} \right\| &= \left\| -\frac{\Gamma\tilde{f}(2^i x)}{720 \cdot 32^{i+1}} \right\| \leq \frac{K'\theta\|2^i x\|^p}{32^{i+1}} \end{aligned}$$

for all  $x \in X \setminus \{0\}$ . Notice, for each  $k = 1, 2, \dots, 5$ , it is clear that

$$\frac{\tilde{f}_k(2^n x)}{2^{kn}} - \frac{\tilde{f}_k(2^{n+m} x)}{2^{k(n+m)}} = \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1} x)}{2^{k(i+1)}} \right)$$

for all  $x \in X$ , and so we easily obtain

$$(2.9) \quad \left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{4K'\theta \|2^i x\|^p}{2^{i+1}},$$

$$(2.10) \quad \left\| \frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m} x)}{4^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_2(2^i x)}{2^i} - \frac{\tilde{f}_2(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{K\theta \|2^i x\|^p}{4^{i+1}},$$

$$(2.11) \quad \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m} x)}{8^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_3(2^i x)}{2^i} - \frac{\tilde{f}_3(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{5K'\theta \|2^i x\|^p}{8^{i+1}},$$

$$(2.12) \quad \left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m} x)}{16^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_4(2^i x)}{2^i} - \frac{\tilde{f}_4(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{K\theta \|2^i x\|^p}{16^{i+1}},$$

$$(2.13) \quad \left\| \frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m} x)}{32^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_5(2^i x)}{2^i} - \frac{\tilde{f}_5(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{K'\theta \|2^i x\|^p}{32^{i+1}}$$

for all  $x \in X \setminus \{0\}$  and  $n, m \in \mathbb{N} \cup \{0\}$ . They imply that

$$(2.14) \quad \left\| \sum_{k=1}^5 \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^5 \frac{\tilde{f}_k(2^{n+m} x)}{2^{k(n+m)}} \right\| \\ \leq \sum_{i=n}^{n+m-1} \left( \frac{4K'}{2^{i+1}} + \frac{K}{4^{i+1}} + \frac{5K'}{8^{i+1}} + \frac{K}{16^{i+1}} + \frac{K'}{32^{i+1}} \right) \theta \|2^i x\|^p$$

for all  $x \in X \setminus \{0\}$  and  $n, m \in \mathbb{N} \cup \{0\}$ . Since  $p < 0$ , the inequalities (2.9)-(2.14) lead us to show that all the sequences  $\left\{ \frac{\tilde{f}_1(2^n x)}{2^n} \right\}, \dots, \left\{ \frac{\tilde{f}_5(2^n x)}{2^{5n}} \right\}$ , and  $\left\{ \sum_{k=1}^5 \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\}$  are Cauchy sequences for all  $x \in X \setminus \{0\}$ . Moreover, since  $Y$  is complete and  $\tilde{f}(0) = 0$ , the sequences  $\left\{ \frac{\tilde{f}_1(2^n x)}{2^n} \right\}, \dots, \left\{ \frac{\tilde{f}_5(2^n x)}{2^{5n}} \right\}$ , and  $\left\{ \sum_{k=1}^5 \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\}$  converge for all  $x \in X$ , too. Hence, for each  $k = 1, 2, \dots, 5$ , we can define the mappings  $F_k, F : X \rightarrow Y$  by

$$F_k(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_k(2^n x)}{2^{kn}}, \quad F(x) := \lim_{n \rightarrow \infty} \sum_{k=1}^5 \frac{\tilde{f}_k(2^n x)}{2^{kn}}$$

for all  $x \in X$ . Now, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.14). Since  $\tilde{f}(x) = \sum_{k=1}^5 \tilde{f}_k(x)$  for all  $x \in X$ , see (2.1), we get the inequality

$$(2.15) \quad \|\tilde{f}(x) - F(x)\| \\ \leq \left( \frac{4K'}{2-2^p} + \frac{K}{4-2^p} + \frac{5K'}{8-2^p} + \frac{K}{16-2^p} + \frac{K'}{32-2^p} \right) \theta \|x\|^p$$

for all  $x \in X$ . By (2.6) and the definition of  $\tilde{f}_1$  in Lemma 2.1, we easily get

$$\|CF_1(x, y)\| = \lim_{n \rightarrow \infty} \left\| \frac{C\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| \\ = \lim_{n \rightarrow \infty} \left\| \frac{256C\tilde{f}_o(2^n x, 2^n y)}{180 \cdot 2^n} - \frac{40C\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{180 \cdot 2^n} + \frac{C\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{180 \cdot 2^n} \right\| \\ \leq \lim_{n \rightarrow \infty} \frac{(256 \cdot 2^{np} + 40 \cdot 2^{(n+1)p} + 2^{(n+2)p}) \theta (\|x\|^p + \|y\|^p)}{180 \cdot 2^n} \\ = 0$$

for all  $x, y \in X \setminus \{0\}$ , since  $p < 0$ . Observe that

$$\begin{aligned} CF_1(x, 0) &= \sum_{i=0}^6 {}_6C_i (-1)^{6-i} F_1(x) = 0 \\ CF_1(0, y) &= \sum_{i=0}^6 {}_6C_i (-1)^{6-i} F_1(iy) = \sum_{i=0}^6 {}_6C_{6-i} (-1)^i F_1((6-i)y) \\ &= \sum_{i=0}^6 {}_6C_i (-1)^{6-i} F_1((6-i)y) = CF_1(6y, -y) = 0 \end{aligned}$$

for all  $x \in X$  and  $y \in X \setminus \{0\}$ . Therefore, we have that  $CF_1(x, y) = 0$  for all  $x, y \in X$ . Similarly we can show that  $CF_2(x, y) = \cdots = CF_5(x, y) = 0$  for all  $x, y \in X$ . Hence we have  $CF(x, y) = \sum_{k=1}^5 CF_k(x, y) = 0$  for all  $x, y \in X$ . Now we use the equation

$$\begin{aligned} C\tilde{f}((1-n)x, nx) &= C\tilde{f}((1-n)x, nx) - CF((1-n)x, nx) \\ &= \sum_{i=0}^6 {}_6C_i (-1)^{6-i} \left( \tilde{f}((1-n)x + inx) - F((1-n)x + inx) \right) \\ &= \tilde{f}((1-n)x) - F((1-n)x) - 6 \left( \tilde{f}(x) - F(x) \right) \\ &\quad + \sum_{i=2}^6 {}_6C_i (-1)^{6-i} \left( \tilde{f}((1-n)x + inx) - F((1-n)x + inx) \right), \end{aligned}$$

together with (2.6), (2.15) and the property  $p < 0$ , to show that

$$\begin{aligned} &6 \|\tilde{f}(x) - F(x)\| \\ &\leq \lim_{n \rightarrow \infty} \|C\tilde{f}((1-n)x, nx)\| + \lim_{n \rightarrow \infty} \left\| \tilde{f}((1-n)x) - F((1-n)x) \right\| \\ &\quad + \sum_{i=2}^6 \lim_{n \rightarrow \infty} \left\| {}_6C_i \left( \tilde{f}(((i-1)n+1)x) - F(((i-1)n+1)x) \right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( |1-n|^p + n^p + M|1-n|^p + M \sum_{i=2}^6 {}_6C_i |(i-1)n+1|^p \right) \theta \|x\|^p \\ &= 0 \end{aligned}$$

for all  $x \in X \setminus \{0\}$ , where  $M := \frac{4K'}{2-2p} + \frac{K}{4-2p} + \frac{5K'}{8-2p} + \frac{K}{16-2p} + \frac{K'}{32-2p}$ . Moreover, since  $\tilde{f}(0) = 0 = F(0)$ , we get  $\tilde{f}(x) = F(x)$  for all  $x \in X$ . Therefore we conclude that,  $Cf(x, y) = C\tilde{f}(x, y) = CF(x, y) = 0$  for all  $x, y \in X$ , and so  $f$  becomes a quintic mapping as desired.  $\square$



### 3. Hyperstability of a general septic functional equation

LEMMA 3.1. For a given mapping  $f : V \rightarrow Y$ , let  $f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12} : V \rightarrow Y$  be the mappings defined by followings;

$$\begin{aligned}
 f_6(x) &:= \frac{1}{N} \begin{vmatrix} f_o(x) & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 \\ f_o(4x) & 8^2 & 32^2 & 128^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 \end{vmatrix} \\
 &= \frac{64(32768f_o(x) - 5376f_o(2x) + 168f_o(4x) - f_o(8x))}{1451520}, \\
 f_7(x) &:= \frac{1}{N'} \begin{vmatrix} f_e(x) & 1 & 1 \\ f_e(2x) & 16 & 64 \\ f_e(4x) & 16^2 & 64^2 \end{vmatrix} = \frac{4(1024f_e(x) - 80f_e(2x) + f_e(4x))}{2880}, \\
 f_8(x) &:= \frac{1}{N} \begin{vmatrix} 1 & f_o(x) & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 \\ 2^2 & f_o(4x) & 32^2 & 128^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 \end{vmatrix} \\
 &= -\frac{84(8192f_o(x) - 4416f_o(2x) + 162f_o(4x) - f_o(8x))}{1451520}, \\
 f_9(x) &:= \frac{1}{N'} \begin{vmatrix} 1 & f_e(x) & 1 \\ 4 & f_e(2x) & 64 \\ 4^2 & f_e(4x) & 64^2 \end{vmatrix} = -\frac{5(256f_e(x) - 68f_e(2x) + f_e(4x))}{2880}, \\
 f_{10}(x) &:= \frac{1}{N} \begin{vmatrix} 1 & 1 & f_o(x) & 1 \\ 2 & 8 & f_o(2x) & 128 \\ 2^2 & 8^2 & f_o(4x) & 128^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 \end{vmatrix} \\
 &= \frac{21(2048f_o(x) - 1296f_o(2x) + 138f_o(4x) - f_o(8x))}{1451520}, \\
 f_{11}(x) &:= \frac{1}{N'} \begin{vmatrix} 1 & 1 & f_e(x) \\ 4 & 16 & f_e(2x) \\ 4^2 & 16^2 & f_e(4x) \end{vmatrix} = \frac{64f_e(x) - 20f_e(2x) + f_e(4x)}{2880} \\
 f_{12}(x) &:= \frac{1}{N} \begin{vmatrix} 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & f_o(8x) \end{vmatrix} \\
 &= -\frac{512f_o(x) - 336f_o(2x) + 42f_o(4x) - f_o(8x)}{1451520}
 \end{aligned}$$

for all  $x \in V$ , where

$$N := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 \\ 2^2 & 8^2 & 32^2 & 128^2 \\ 2^3 & 8^3 & 32^3 & 128^3 \end{vmatrix} \quad \text{and} \quad N' := \begin{vmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 4^2 & 16^2 & 64^2 \end{vmatrix}.$$

Then we have

$$(3.1) \quad f(x) = f_o(x) + f_e(x) = \sum_{i=6}^{12} f_i(x)$$

for all  $x \in V$ .

**Proof.** We note that  $N, N' \neq 0$ . The uniqueness of solution (stated in Cramer's rule) implies that the family  $\{f_6(x), f_8(x), f_{10}(x), f_{12}(x)\}$  is the only solution to the system of nonhomogeneous linear equations

$$(3.2) \quad \begin{cases} f_6(x) + f_8(x) + f_{10}(x) + f_{12}(x) = f_o(x), \\ 2f_6(x) + 8f_8(x) + 32f_{10}(x) + 128f_{12}(x) = f_o(2x), \\ 2^2f_6(x) + 8^2f_8(x) + 32^2f_{10}(x) + 128^2f_{12}(x) = f_o(4x), \\ 2^3f_6(x) + 8^3f_8(x) + 32^3f_{10}(x) + 128^3f_{12}(x) = f_o(8x) \end{cases}$$

for all  $x \in V$ . Therefore, it follows from the first equation of (3.2) that  $f_o(x) = f_6(x) + f_8(x) + f_{10}(x) + f_{12}(x)$  for all  $x \in V$ . In the similar way, we get  $f_e(x) = f_7(x) + f_9(x) + f_{11}(x)$  for all  $x \in V$ . So we have obtained the equation (3.1)

□

In this section, we use the following definitions.

**DEFINITION 3.2.** For a given mapping  $f : V \rightarrow Y$ , we define the mappings  $Df : V^2 \rightarrow Y$  and  $\tilde{\Gamma}f, \tilde{\Delta}f : V \rightarrow Y$  as

$$Df(x, y) := \sum_{i=0}^8 {}_8C_i (-1)^{8-i} f(x + iy),$$

$$\begin{aligned} \tilde{\Gamma}f(x) &:= Df_o(16x, -2x) + 8Df_o(-2x, 2x) + 36Df_o(-4x, 2x) \\ &\quad + 120Df_o(-6x, 2x) + 160Df_o(8x, -x) + 1280Df_o(-x, x) \\ &\quad + 4032Df_o(-2x, x) + 5376Df_o(-3x, x), \end{aligned}$$

$$\begin{aligned} \tilde{\Delta}f(x) &:= Df_e(8x, -x) + 8Df_e(-x, x) + 36Df_e(-2x, x) + 120Df_e(-3x, x) \\ &\quad + 123Df_e(-4x, x) \end{aligned}$$

for all  $x, y \in V$ .

By laborious computation we can get the equalities in the following lemma.

LEMMA 3.3. *Let  $f : V \rightarrow Y$  be an arbitrarily given mapping. Then the equalities*

$$(3.3) \quad \begin{aligned} \tilde{\Gamma}\tilde{f}(x) = \tilde{\Gamma}f(x) = f_o(16x) - 170f_o(8x) \\ + 5712f_o(4x) - 43520f_o(2x) + 65536f_o(x), \end{aligned}$$

$$(3.4) \quad \tilde{\Delta}\tilde{f}(x) = \tilde{f}_e(8x) - 84\tilde{f}_e(4x) + 1344\tilde{f}_e(2x) - 4096\tilde{f}_e(x)$$

hold for all  $x \in V$ .

We show the hyperstability of the septic functional equation  $Df(x, y) = 0$  by following;

THEOREM 3.4. *Let  $p < 0$  and  $\theta > 0$  be real numbers. Suppose that  $f : X \rightarrow Y$  is a mapping such that*

$$(3.5) \quad \|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X \setminus \{0\}$ . Then

$$Df(x, y) = 0$$

for all  $x, y \in X$ , and thus the mapping  $f$  is a generalized polynomial of degree at most 7.

**Proof.** Observe that  $D\tilde{f}(x, y) = Df(x, y)$  and we have

$$(3.6) \quad \|D\tilde{f}(x, y)\|, \|D\tilde{f}_o(x, y)\|, \|D\tilde{f}_e(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x \in X \setminus \{0\}$  by (3.5). By the definitions of  $\tilde{\Gamma}\tilde{f}$  and  $\tilde{\Delta}\tilde{f}$  in Definition 3.1, it follows that

$$(3.7) \quad \|\tilde{\Gamma}\tilde{f}(x)\| \leq 1451520\tilde{K}'\theta\|x\|^p,$$

$$(3.8) \quad \|\tilde{\Delta}\tilde{f}(x)\| \leq 2880\tilde{K}\theta\|x\|^p$$

for all  $x \in X \setminus \{0\}$ , where

$$\tilde{K}' := \frac{16^p + 160 \cdot 8^p + 120 \cdot 6^p + 36 \cdot 4^p + 5376 \cdot 3^p + 4205 \cdot 2^p + 12128}{1451520},$$

$$\tilde{K} := \frac{8^p + 123 \cdot 4^p + 120 \cdot 3^p + 36 \cdot 2^p + 296}{2880}.$$

From the definitions of  $f_6, f_7, \dots, f_{12}$  in Lemma 3.1, together with the properties (3.3), (3.4), (3.7) and (3.8), we obtain the inequalities

$$\begin{aligned} \left\| \frac{\tilde{f}_6(2^i x)}{2^i} - \frac{\tilde{f}_6(2^{i+1} x)}{2^{i+1}} \right\| &= \left\| \frac{64\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 2^{i+1}} \right\| \leq \frac{64\tilde{K}'\theta\|2^i x\|^p}{2^{i+1}}, \\ \left\| \frac{\tilde{f}_7(2^i x)}{4^i} - \frac{\tilde{f}_7(2^{i+1} x)}{4^{i+1}} \right\| &= \left\| -\frac{4\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 4^{i+1}} \right\| \leq \frac{4\tilde{K}\theta\|2^i x\|^p}{4^{i+1}}, \\ \left\| \frac{\tilde{f}_8(2^i x)}{8^i} - \frac{\tilde{f}_8(2^{i+1} x)}{8^{i+1}} \right\| &= \left\| \frac{-84\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 8^{i+1}} \right\| \leq \frac{84\tilde{K}'\theta\|2^i x\|^p}{8^{i+1}}, \\ \left\| \frac{\tilde{f}_9(2^i x)}{16^i} - \frac{\tilde{f}_9(2^{i+1} x)}{16^{i+1}} \right\| &= \left\| \frac{5\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 16^{i+1}} \right\| \leq \frac{5\tilde{K}\theta\|2^i x\|^p}{16^{i+1}}, \\ \left\| \frac{\tilde{f}_{10}(2^i x)}{32^i} - \frac{\tilde{f}_{10}(2^{i+1} x)}{32^{i+1}} \right\| &= \left\| \frac{21\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 32^{i+1}} \right\| \leq \frac{21\tilde{K}'\theta\|2^i x\|^p}{32^{i+1}}, \\ \left\| \frac{\tilde{f}_{11}(2^i x)}{64^i} - \frac{\tilde{f}_{11}(2^{i+1} x)}{64^{i+1}} \right\| &= \left\| -\frac{\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 64^{i+1}} \right\| \leq \frac{\tilde{K}\theta\|2^i x\|^p}{64^{i+1}}, \\ \left\| \frac{\tilde{f}_{12}(2^i x)}{128^i} - \frac{\tilde{f}_{12}(2^{i+1} x)}{128^{i+1}} \right\| &= \left\| -\frac{\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 128^{i+1}} \right\| \leq \frac{\tilde{K}'\theta\|2^i x\|^p}{128^{i+1}} \end{aligned}$$

for all  $x \in X \setminus \{0\}$ . For each  $k = 6, 7, \dots, 12$ , it is clear that

$$\frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} - \frac{\tilde{f}_k(2^{n+m} x)}{2^{(k-5)(n+m)}} = \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_k(2^i x)}{2^{(k-5)i}} - \frac{\tilde{f}_k(2^{i+1} x)}{2^{(k-5)(i+1)}} \right)$$

for all  $x \in X$ , so we obtain that

$$(3.9) \quad \left\| \frac{\tilde{f}_6(2^n x)}{2^n} - \frac{\tilde{f}_6(2^{n+m} x)}{2^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_6(2^i x)}{2^i} - \frac{\tilde{f}_6(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{64\tilde{K}'\theta\|2^i x\|^p}{2^{i+1}},$$

$$(3.10) \quad \left\| \frac{\tilde{f}_7(2^n x)}{4^n} - \frac{\tilde{f}_7(2^{n+m} x)}{4^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_7(2^i x)}{2^i} - \frac{\tilde{f}_7(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{4\tilde{K}\theta\|2^i x\|^p}{4^{i+1}},$$

$$(3.11) \quad \left\| \frac{\tilde{f}_8(2^n x)}{8^n} - \frac{\tilde{f}_8(2^{n+m} x)}{8^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_8(2^i x)}{2^i} - \frac{\tilde{f}_8(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{84\tilde{K}'\theta\|2^i x\|^p}{8^{i+1}},$$

$$(3.12) \quad \left\| \frac{\tilde{f}_9(2^n x)}{16^n} - \frac{\tilde{f}_9(2^{n+m} x)}{16^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_9(2^i x)}{2^i} - \frac{\tilde{f}_9(2^{i+1} x)}{2^{i+1}} \right\|,$$

$$\leq \sum_{i=n}^{n+m-1} \frac{5\tilde{K}\theta\|2^i x\|^p}{16^{i+1}}$$

$$(3.13) \quad \left\| \frac{\tilde{f}_{10}(2^n x)}{32^n} - \frac{\tilde{f}_{10}(2^{n+m} x)}{32^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{10}(2^i x)}{2^i} - \frac{\tilde{f}_{10}(2^{i+1} x)}{2^{i+1}} \right\|,$$

$$\leq \sum_{i=n}^{n+m-1} \frac{21\tilde{K}'\theta\|2^i x\|^p}{32^{i+1}}$$

$$(3.14) \quad \left\| \frac{\tilde{f}_{11}(2^n x)}{64^n} - \frac{\tilde{f}_{11}(2^{n+m} x)}{64^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{11}(2^i x)}{2^i} - \frac{\tilde{f}_{11}(2^{i+1} x)}{2^{i+1}} \right\|,$$

$$\leq \sum_{i=n}^{n+m-1} \frac{\tilde{K}\theta\|2^i x\|^p}{64^{i+1}}$$

$$(3.15) \quad \left\| \frac{\tilde{f}_{12}(2^n x)}{128^n} - \frac{\tilde{f}_{12}(2^{n+m} x)}{128^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{12}(2^i x)}{2^i} - \frac{\tilde{f}_{12}(2^{i+1} x)}{2^{i+1}} \right\|$$

$$\leq \sum_{i=n}^{n+m-1} \frac{\tilde{K}'\theta\|2^i x\|^p}{128^{i+1}}$$

for all  $x \in X \setminus \{0\}$  and  $n, m \in \mathbb{N} \cup \{0\}$ . They imply that

$$(3.16) \quad \left\| \sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} - \sum_{k=6}^{12} \frac{\tilde{f}_k(2^{n+m} x)}{2^{(k-5)(n+m)}} \right\| \leq \sum_{i=n}^{n+m-1} \left( \frac{64\tilde{K}'}{2^{i+1}} + \frac{4\tilde{K}}{4^{i+1}} \right. \\ \left. + \frac{84\tilde{K}'}{8^{i+1}} + \frac{5\tilde{K}}{16^{i+1}} + \frac{21\tilde{K}'}{32^{i+1}} + \frac{\tilde{K}}{64^{i+1}} + \frac{\tilde{K}'}{128^{i+1}} \right) \theta \|2^i x\|^p$$

for all  $x \in X \setminus \{0\}$  and  $n, m \in \mathbb{N} \cup \{0\}$ . Since  $p < 0$ , the inequalities (3.9)-(3.16) lead us to prove that all the sequences  $\left\{ \frac{\tilde{f}_6(2^n x)}{2^n} \right\}, \dots, \left\{ \frac{\tilde{f}_{12}(2^n x)}{2^{7n}} \right\}$ , and  $\left\{ \sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} \right\}$  are Cauchy sequences for all  $x \in X \setminus \{0\}$ . Moreover, since  $Y$  is complete and  $\tilde{f}(0) = 0$ , the sequences

$$\left\{ \frac{\tilde{f}_6(2^n x)}{2^n} \right\}, \dots, \left\{ \frac{\tilde{f}_{12}(2^n x)}{2^{7n}} \right\}, \quad \text{and} \quad \left\{ \sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} \right\}$$

converge for all  $x \in X$ , too. Hence, for each  $k = 6, 7, \dots, 12$ , we can define mappings  $F_k, \hat{F} : X \rightarrow Y$  by

$$F_k(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}}, \\ \hat{F}(x) := \lim_{n \rightarrow \infty} \sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}}$$

for all  $x \in X$ . Now, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (3.16), by

$$\tilde{f}(x) = \sum_{k=6}^{12} \tilde{f}_k(x)$$

in (3.1), we get the inequality

$$(3.17) \quad \|\tilde{f}(x) - \hat{F}(x)\| \leq \left( \frac{64\tilde{K}'}{2-2^p} + \frac{4\tilde{K}}{4-2^p} + \frac{84\tilde{K}'}{8-2^p} + \frac{5\tilde{K}}{16-2^p} \right. \\ \left. + \frac{21\tilde{K}'}{32-2^p} + \frac{\tilde{K}}{64-2^p} + \frac{\tilde{K}'}{128-2^p} \right) \theta \|x\|^p$$

for all  $x \in X$ . By (3.6), we easily have that

$$\begin{aligned}
\|DF_6(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_6(2^n x, 2^n y)}{2^n} \right\| \\
&= \lim_{n \rightarrow \infty} \left\| \frac{32768D\tilde{f}_o(2^n x, 2^n y)}{2835 \cdot 8 \cdot 2^n} - \frac{5376D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{2835 \cdot 8 \cdot 2^n} \right. \\
&\quad \left. + \frac{168D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{2835 \cdot 8 \cdot 2^n} - \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{2835 \cdot 8 \cdot 2^n} \right\| \\
&\leq \lim_{n \rightarrow \infty} \frac{32768 \cdot 2^{np}\theta(\|x\|^p + \|y\|^p)}{2835 \cdot 8 \cdot 2^n} + \lim_{n \rightarrow \infty} \frac{5376 \cdot 2^{(n+1)p}\theta(\|x\|^p + \|y\|^p)}{2835 \cdot 8 \cdot 2^n} \\
&\quad + \lim_{n \rightarrow \infty} \frac{168 \cdot 2^{(n+2)p}\theta(\|x\|^p + \|y\|^p)}{2835 \cdot 8 \cdot 2^n} + \lim_{n \rightarrow \infty} \frac{2^{(n+3)p}\theta(\|x\|^p + \|y\|^p)}{2835 \cdot 8 \cdot 2^n} \\
&= 0
\end{aligned}$$

for all  $x, y \in X \setminus \{0\}$ . Observe that

$$DF_6(x, 0) = \sum_{i=0}^8 {}_8C_i(-1)^{8-i}F_6(x) = 0$$

for all  $x \in X$ , as well as

$$\begin{aligned}
DF_6(0, y) &= \sum_{i=0}^8 {}_8C_i(-1)^{8-i}F_6(iy) = \sum_{i=0}^8 {}_8C_{8-i}(-1)^iF_6((8-i)y) \\
&= \sum_{i=0}^8 {}_8C_i(-1)^{8-i}F_6((8-i)y) = DF_6(8y, -y) \\
&= 0
\end{aligned}$$

for all  $y \in X \setminus \{0\}$ . Hence we can say that  $DF_6(x, y) = 0$  for all  $x, y \in X$ . Similarly we get  $DF_7(x, y) = \dots = DF_{12}(x, y) = 0$  for all  $x, y \in X$ . Therefore, we obtain that

$$D\hat{F}(x, y) = \sum_{k=6}^{12} DF_k(x, y) = 0$$

for all  $x, y \in X$ , i.e., the mapping  $\hat{F}$  is a septic mapping. Now, together with the equation

$$\begin{aligned} D\tilde{f}((1-n)x, nx) &= D\tilde{f}((1-n)x, nx) - D\hat{F}((1-n)x, nx) \\ &= \sum_{i=0}^8 {}_8C_i(-1)^{8-i} \left( \tilde{f}((1-n)x + inx) - \hat{F}((1-n)x + inx) \right) \\ &= \tilde{f}((1-n)x) - \hat{F}((1-n)x) - 8 \left( \tilde{f}(x) - \hat{F}(x) \right) \\ &\quad + \sum_{i=2}^8 {}_8C_i(-1)^{8-i} \left( \tilde{f}((1-n)x + inx) - \hat{F}((1-n)x + inx) \right) \end{aligned}$$

for all  $x \in X$ , it follows that

$$\begin{aligned} 8\|\tilde{f}(x) - \hat{F}(x)\| &\leq \lim_{n \rightarrow \infty} \left\| D\tilde{f}((1-n)x, nx) \right\| + \lim_{n \rightarrow \infty} \left\| \tilde{f}((1-n)x) - \hat{F}((1-n)x) \right\| \\ &\quad + \sum_{i=2}^8 \lim_{n \rightarrow \infty} \left\| {}_8C_i \left( \tilde{f}(((i-1)n+1)x) - \hat{F}(((i-1)n+1)x) \right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( |1-n|^p + n^p + \tilde{M}|1-n|^p + \tilde{M} \sum_{i=2}^8 {}_8C_i |(i-1)n+1|^p \right) \theta \|x\|^p \\ &= 0 \end{aligned}$$

for all  $x \in X \setminus \{0\}$ , by (3.6), (3.17) and the property  $p < 0$ , where

$$\begin{aligned} \tilde{M} &:= \frac{64K'}{2-2^p} + \frac{4K}{4-2^p} + \frac{84K'}{8-2^p} + \frac{5K}{16-2^p} \\ &\quad + \frac{21K'}{32-2^p} + \frac{K}{64-2^p} + \frac{K'}{128-2^p}. \end{aligned}$$

Since  $\tilde{f}(0) = 0 = F(0)$ , we get  $\tilde{f}(x) = F(x)$  for all  $x \in X$ . Therefore we conclude that,

$$Df(x, y) = D\tilde{f}(x, y) = DF(x, y) = 0$$

for all  $x, y \in X$ , as we desired. □



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